TRINITY COLLEGE

IN NOMINE DOMINI

Semester Two Examination, 2018

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	51	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

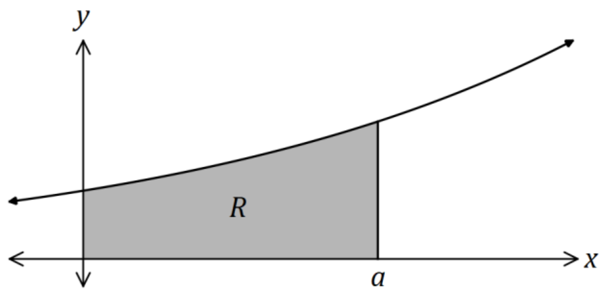
Section One: Calculator-free **35% (51 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

The shaded region R, shown on the graph below, is bounded by the curve $y = e^{4x}$ and the lines $y = 0,\ x = 0$ and $x = a$.



(a) Calculate the area of R in terms of a. (3 marks)

Solution
$R = \int_0^a e^{4x}\,dx$ $= \left[\frac{e^{4x}}{4}\right]_0^a$ $= \frac{e^{4a}}{4} - \frac{e^0}{4} = \frac{e^{4a}}{4} - \frac{1}{4}$
Specific behaviours
✓ writes correct integral   ✓ antidifferentiates correctly   ✓ substitutes and simplifies

(b) Determine the value of a for which the area of R is 20 square units. (3 marks)

Solution
$\frac{e^{4a}}{4} - \frac{1}{4} = 20 \Rightarrow e^{4a} = 81$ $4a = \ln 81 \Rightarrow a = \frac{1}{4}\ln 81$ $a = \ln \sqrt[4]{81} = \ln 3$
Specific behaviours
✓ isolates e^{4a} term   ✓ uses logs to obtain expression for a ✓ simplifies

Question 2 **(5 marks)**

(a) Simplify $\log_2(16) \div \log_5(125^2)$. (2 marks)

Solution
$\dfrac{\log_2 2^4}{\log_5 5^6} = \dfrac{4}{6} = \dfrac{2}{3}$
Specific behaviours
✓ expresses as powers of log bases ✓ simplifies

(b) Solve the equation $\ln(4 - x) + \ln 2 = 2 \ln x$. (3 marks)

Solution
$\ln(8 - 2x) = \ln x^2$ $x^2 + 2x - 8 = 0$ $(x + 4)(x - 2) = 0$ $x = -4$ or $x = 2$ But from equation, $0 < x < 4$ $\therefore x = 2$
Specific behaviours
✓ writes both sides as single logs ✓ factorises quadratic ✓ identifies just one solution

Question 3 **(7 marks)**

The graph of $y = 3xe^{2x}$ has one stationary point.

(a) Determine the x-coordinate of the stationary point of the graph. (3 marks)

Solution
$$\frac{dy}{dx} = 6xe^{2x} + 3e^{2x}$$ $$\frac{dy}{dx} = 0 \Rightarrow 3e^{2x}(2x+1) = 0$$ $$\text{Hence stationary points when } x = \frac{-1}{2}$$
Specific behaviours
✓ uses product rule correctly ✓ factorises derivative ✓ states x-coordinate

(b) Use the second derivative test to determine whether the stationary point from (a) is a local maximum or a local minimum and state the coordinates of this point. (4 marks)

Solution
$$\frac{dy}{dx} = 3e^{2x}(2x+1)$$ $$\frac{d^2y}{dx^2} = 3e^{2x}(2) + (2x+1)6e^{2x}$$ $$x = \frac{-1}{2} \Rightarrow \frac{d^2y}{dx^2} = 3e^{-1}(2) + 0 \Rightarrow \text{Minimum}$$ $$\text{Hence minimum at } x = \frac{-1}{2}$$ $$\text{Min at } \left(\frac{-1}{2}, \frac{-3}{2e}\right)$$
Specific behaviours
✓ second derivative using product rule ✓ substitutes an x-value ✓ interprets sign of second derivative ✓ deduces required x-value and states coordinates

Question 4 **(6 marks)**

The random variable X has probability density function

$$f(x) = \begin{cases} k(2x-1)^3, & 0.5 \le x \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine the value of the constant k. (4 marks)

Solution
$$k\int_{0.5}^{2}(2x-1)^3\,dx = 1$$ $$\int_{0.5}^{2}(2x-1)^3\,dx = \left[\frac{(2x-1)^4}{8}\right]_{0.5}^{2}$$ $$= \frac{3^4}{8} - \frac{0^4}{8} = \frac{81}{8}$$ $$\frac{81}{8}k = 1 \Rightarrow k = \frac{8}{81}$$
Specific behaviours
✓ writes integral with correct limits ✓ integrates correctly ✓ equates integral to 1 ✓ correct value of k

(b) Write down the cumulative distribution function $F(t) = P(X \le t)$ for $0.5 \le t \le 2$ and hence determine $P(X \le 1)$. (2 marks)

Solution
$$F(t) = \frac{8}{81}\int_{0.5}^{t}(2x-1)^3\,dx = \frac{1}{81}(2t-1)^4$$ $$F(1) = \frac{1}{81}$$
Specific behaviours
✓ correct $F(t)$ ✓ correct probability

Question 5 **(6 marks)**

(a) Determine the anti-derivative of $\dfrac{\cos(4x)}{7+\sin(4x)}$. (2 marks)

Solution
$f(x)=\dfrac{1}{4}\int \dfrac{4\cos 4x}{7+\sin 4x}\,dx$ $=\dfrac{1}{4}\ln(7+\sin 4x)+c$
Specific behaviours
✓ writes in form $f'(x) \div f(x)$ ✓ correct integral and constant

(b) (i) Determine $f'(x)$ when $f(x)=4x\ln(3x)$. (2 marks)

Solution
$f'(x)=4\times\ln(3x)+4x\times\dfrac{3}{3x}$ $=4\ln(3x)+4$
Specific behaviours
✓ uses product rule correctly ✓ differentiates log term correctly

(ii) Hence, or otherwise, evaluate $\int_{\frac{1}{3}}^{2}(4\ln(3x)+4)\,dx$. (2 marks)

Solution
$[4x\ln(3x)]_{1/3}^{2}$ $=8\ln 6-\dfrac{4}{3}\ln 1$ $=8\ln 6$
Specific behaviours
✓ antiderivative ✓ evaluates correctly

Question 6 **(7 marks)**

A closed cylindrical can of radius r cm has a volume of 250π cm^3.

(a) Show that the total surface area, A cm^2, of this can is given by $A=\dfrac{500\pi}{r}+2\pi r^2$.

(2 marks)

Solution
$V=\pi r^2 h$ $250\pi=\pi r^2 h \;\Rightarrow\; h=\dfrac{250}{r^2}$ $A=2\pi r^2+2\pi rh$ $\quad=2\pi r^2+2\pi r\dfrac{250}{r^2}$ $\quad=\dfrac{500\pi}{r}+2\pi r^2$
Specific behaviours
✓ determines an expression for h ✓ substitutes into area formula correctly

(b) Determine the minimum possible surface area of the can and the radius and height required to achieve this optimum area. (5 marks)

Solution
$\dfrac{dA}{dr}=-\dfrac{500\pi}{r^2}+4\pi r$ $-\dfrac{500\pi}{r^2}+4\pi r=0$ $r^3=125 \;\Rightarrow\; r=5$ cm $h=\dfrac{250}{5^2}=10$ cm $A=\dfrac{500\pi}{5}+2\pi\times 5^2$ $A=150\pi$
Specific behaviours
✓ differentiates correctly   ✓ equates equal to 0 ✓ solves for r ✓ solves for h ✓ calculates area

Question 7 **(6 marks)**

The time, t years, to repay a loan of $\$180\,000$ at 4% interest with monthly repayments of x dollars can be approximated by

$$t = 25\ln\left(\frac{x}{x-600}\right), \quad x > 600$$

(a) Determine the expression for the time to repay the loan when the monthly repayment is $\$750$. (1 mark)

Solution
$t = 25\ln\left(\frac{750}{150}\right)$ years
Specific behaviours
✓ substitutes

(b) Use the increments formula to estimate the time saved in repaying the loan if the monthly repayment of $\$750$ is increased by 2%. (5 marks)

Solution
$t = 25\ln x - 25\ln(x-600)$ $\frac{dt}{dx} = \frac{25}{x} - \frac{25}{x-600}$ $\left.\frac{dt}{dx}\right\|_{x=750} = \frac{25}{750} - \frac{25}{150} = -\frac{100}{750}$ $\delta x = 750 \times 0.02$ $\delta t \approx -\frac{100}{750} \times 750 \times 0.02 \approx -2$ Time saved is 2 years
Specific behaviours
✓ uses log laws to simplify t ✓ correct derivative ✓ evaluates derivative ✓ indicates value of δx ✓ uses increments formula and states time saved

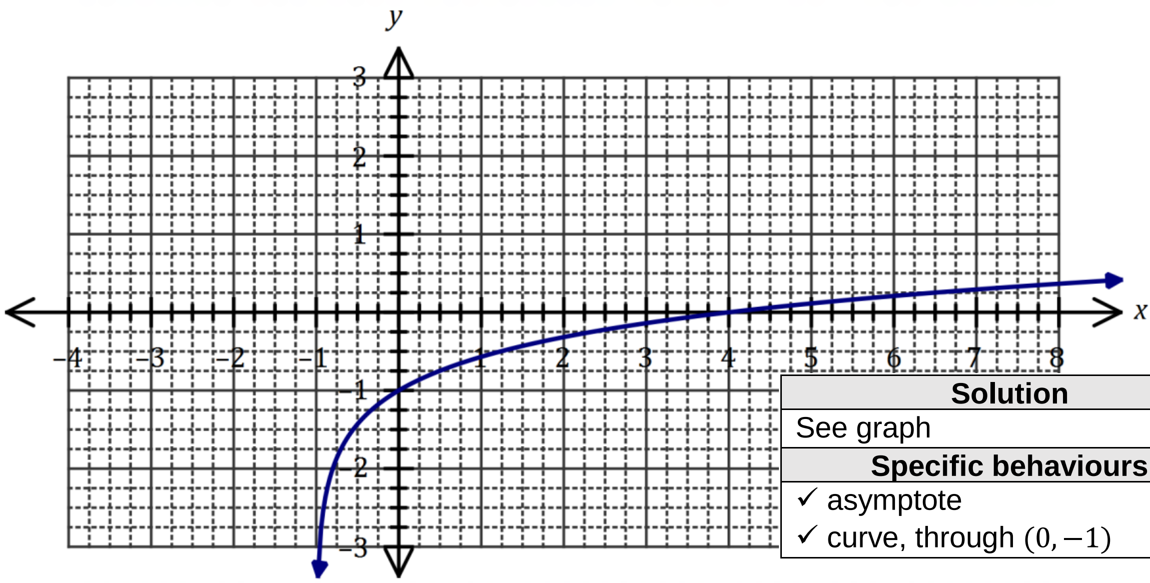
Question 8 **(8 marks)**

(a) Determine the coordinates of the x-intercept and y-intercept of the graph of $y = \log_5(x+1) - 1$. (2 marks)

Solution
$0 = \log_5(x+1) - 1 \Rightarrow 5 = x + 1 \Rightarrow x = 4$ $y = \log_5(0+1) - 1 \Rightarrow y = 0 - 1 \Rightarrow y = -1$
Specific behaviours
✓ simplifies log term to -2 ✓ states coordinates of intercept

(b) Sketch the graph of $y = \log_5(x+1) - 1$ on the axes below, clearly showing the location of all asymptotes and axes intercepts.

(2 marks)



Solution
See graph
Specific behaviours
✓ asymptote ✓ curve, through $(0, -1)$

(c) The graph of $y = \log_a(x+a)$, where $a > 1$, passes through $(15.75, 2)$. Determine the coordinates of the root of the graph. (4 marks)

Solution
$2 = \log_a(15.75 + a) \Rightarrow a^2 - a - 15.75 = 0$ $a = \dfrac{1 \pm \sqrt{1 + 4(15.75)}}{2}$ $= \dfrac{1 \pm 8}{2}$ $a = 4.5, \quad (a > 1)$ $\text{Hence root at } (-3.5, 0)$
Specific behaviours
✓✓ forms quadratic equation ✓ solves for a ✓ states coordinates of root

End of questions

Supplementary page

Question number: _________